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ENUMERATING FUZZY SWITCHING FUNCTIONS AND FREE KLEENE ALGEBRAS

JOEL BERMAN
Mathematics Department, University of Illinois at Chicago, Chicago, IL 60680, U.S.A.

and

MASAO MUKAIDONO
Faculty of Engineering, Meiji University, Kawasaki 214, Japan

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Abstract— We investigate the free Kleene algebra on n free generators, FK(n). This algebra is essentially the same as the set of n-variable fuzzy switching functions. The cardinality of FK(n) has been previously determined only for $n = 1, 2$, and 3. We present a general method for enumeration problems of this kind. We show FK(4) has 160,297,985,276 elements. The final section contains new upper and lower bounds for the cardinality of FK(n). We also consider the number of regular functions and show that there are 11,197,129,615, and 430,904,428,717 such functions in 1, 2, 3, and 4 variables.

INTRODUCTION

It is now well understood that the two element Boolean algebra and its equational theory is an important topic for logicians, algebraists, computer scientists and electrical engineers. There is also a three element algebra, a generalization of Boolean algebra frequently called a Kleene algebra, that is of interest to these same groups of people. That this three element algebra and its corresponding equational theory is studied by workers in these diverse areas is not well known—even to some of these researchers. As a result there has been a good deal of duplication of effort in the literature and a consequent lack of standardization of nomenclature and notation for this subject.

In this paper we first briefly survey how Kleene algebras arise in three areas: lattice theory, mathematical logic, and fuzzy set theory. We consider a special problem, that of describing the free Kleene algebra on n free generators. We show how this problem masquerades in different guises in these areas. Section 2 contains a general method for dealing with these free objects. In Section 3 we show that the free Kleene algebra on 4 generators has 160,297,985,276 elements and we provide a manageable decomposition of it. This result is new and it extends previous work for 1, 2, and 3 free generators. Section 4 shows that the general method given in Section 2 is indeed general. We use it to compute free objects in some other related algebraic systems. The final section of the paper contains new upper and lower bounds on the size of the free Kleene algebra on n generators.

1. MOTIVATING PROBLEMS

Our first point of view is obtained by tinkering with a common axiomization of Boolean algebras. A Boolean algebra is a set A with two binary operations $+$ and $\cdot$, two constants 0 and 1, and a unary operation $\bar{}$; the system $\langle A; +, \cdot, -, 0, 1\rangle$ must be such that $\langle A; +, \cdot, 0, 1\rangle$ satisfies the axioms δ of distributive lattices with 0 and 1, and the operation $\bar{}$ must obey the laws:

$$\bar{\bar{x}} = x$$

$$\overline{x+y} = \bar{x}\bar{y} \qquad \overline{xy} = \bar{x} + \bar{y}$$

$$x + \bar{x} = 1 \qquad x\bar{x} = 0.$$

That is, $^{-}$ is a dual lattice automorphism of period 2, in which $\bar{x}$ is the lattice complement of x. Let β denote this set of axioms for Boolean algebra.

We eliminate from β the laws $x + \bar{x} = 1$ and $x\bar{x} = 0$. What change, if any, takes place? The algebra of least cardinality which satisfies all the laws of $\beta - \{x + \bar{x} = 1, x\bar{x} = 0\}$, but not β, is the three element lattice $0 < h < 1$ in which $\bar{0} = 1$, $\bar{1} = 0$ and $\bar{h} = h$. Call this algebra K. K satisfies another law: $x\bar{x}(y + \bar{y}) = x\bar{x}$, and it can be shown that this law is not a consequence of $\beta - \{x + \bar{x} = 1,\ x\bar{x} = 0\}$. Next, let $\kappa = (\beta - \{x + \bar{x} = 1, x\bar{x} = 0\}) \cup \{x\bar{x}(y + \bar{y}) = x\bar{x}\}$. It is known that the family of all equations that are true in the three element algebras K is precisely the same as the set of equations that can be derived from the set κ. For details see Kalman[1] or Balbes and Dwinger[2]. Let $\underset{\sim}{K}$ be the equational class generated by K. That is, $\underset{\sim}{K}$ is the class of all algebras having as operations $+$, $\cdot$, 0, 1, $-$ and these operations satisfy all the laws of κ. This equational class $\underset{\sim}{K}$, generated by the 3-element algebra K viewed as a formal relaxation of the laws of Boolean algebra, has been studied by various workers in the last 25 years[1–10].

In these studies of the equational class $\underset{\sim}{K}$, an algebra frequently considered is FK(n), the free algebra on n free generators. Roughly speaking, FK(n) is the most general algebra in $\underset{\sim}{K}$ that can be generated from n elements; a more precise definition in terms of universal mapping properties together with a discussion of the usefulness of free algebras is found say in Grätzer[52], Cohn[12], or Balbes and Dwinger[2].

Problem 1. For any n, describe FK(n). In particular, what is the cardinality of FK(n)?

Our next approach is derived from Kleene ([13], pp. 332–340). Let $Q(m)$ denote a partial recursive predicate, that is, there is an algorithm to decide for all whole numbers m in the domain of Q, whether or not the predicate Q is true for m; for m not in the domain of Q, $Q(m)$ is undefined. Thus, for a given m, there are three possibilities for $Q(m)$: true (t), false (f), and undefined (u). Next, a propositional logic can be constructed for the connectives "and ", "or", and "not"; denoted $\wedge$, $\vee$, $-$ respectively. Kleene argues that these connectives should have the following behaviour:

$\wedge$	t	u	f
t	t	u	f
u	u	u	f
f	f	f	f

$\vee$	t	u	f
t	t	t	t
u	t	u	u
f	t	u	f

$-$	t	u	f
	f	u	t

From these rules compound predicates such as $(Q_1 \vee \bar{Q}_2) \wedge \overline{(Q_2 \vee Q_3)}$ can be built from the three predicates Q_1, Q_2, and Q_3. Finally, truth tables can be constructed to determine how such a predicate behaves for a particular m and values $Q_1(m)$, $Q_2(m)$, and $Q_3(m)$. Note these truth tables, if built from n predicates, have 3^n rows, and hence there are at most 3^{3^n} possible truth tables. It is easily seen that not all such truth tables arise if only $\wedge$, $\vee$, and $-$ are used.

Problem 2. Describe all the distinct truth tables that can be built from n predicates using $\wedge$, $\vee$, $-$. In particular, how many such truth tables are there?

Note that this problem is also the same as describing the Lindenbaum–Tarski algebra for this logical system. For details see, for example, Rasiowa[14].

Our final approach is from the theory of fuzzy sets, e.g. Zadeh[15], Lee and Chang[16], Kandel[53], Mukaidono[18]. Let x_i denote a fuzzy variable, that is x_i takes values from the real unit interval [0, 1]. Consider the three operations $\wedge$, $\vee$, and $-$ defined pointwise on real-valued functions f and g by $f \wedge g = \min\{f, g\}$, $f \vee g = \max\{f, g\}$ and $\bar{f} = 1 - f$. A fuzzy switching function in n variables is any function that can be built from the n fuzzy variables $x_1, x_2, \ldots, x_n$ and the two constant functions 0 and 1, by using only a finite number of applications of $\wedge$, $\vee$, and $-$. For $n = 1$ there are exactly 6 fuzzy switching functions: 0, 1, x_1, $\bar{x}_1$, $x_1 \wedge \bar{x}_1$, $x_1 \vee \bar{x}_1$. If f is a fuzzy switching function, then f:$[0, 1]^n \to [0, 1]$. If z is any n-tuple in $\{0, 1\}^n$, then $f(z) \in \{0, 1\}$. Similarly if $z \in \{0, 1/2, 1\}^n$, then $f(z) \in \{0, 1/2, 1\}$. Crucial for our approach to fuzzy switching functions in the result of Preparata and Yeh[19]: Any fuzzy switching function f is uniquely determined by its behaviour on $\{0, 1/2, 1\}^n$. Note the set $\{0, 1/2, 1\}$ with the operations $\wedge$, $\vee$, $-$ and the constants 0 and 1 is isomorphic to the algebra K.

Further background on fuzzy switching functions is in Davio, Deschamps and Thayse[20] and Kandel and Lee[21]. An early pioneering work connecting the algebra K to switching functions Goto[22]. Also, Muller[23] briefly discusses such functions.

Problem 3. Describe all fuzzy switching functions in n variables. In particular, how may such functions are there?

In each of Problems 1, 2, and 3 there is the same 3 element system and hence the three problems have basically the same solutions. Note in Problem 2, the two truth tables that are all t or all f should be included so that this problem is exactly equivalent to Problems 1 and 3.

The lattice theoretic point of view towards the class of Kleene algebras is first explicitly found in Kalman[1] and Bialnicki–Birula and Rasiowa[3]. In Kalman's paper the 3 element algebra is called $\mathscr{D}_3$ and K is called the class of *normal i lattices*.

The approach in terms of logic goes back to Kleene[24] and this 3-valued logic is found as a fragment of other logical systems: e.g. Moisil[25], Rasiowa[14] or the Lukasiewicz and Post algebras as in Balbes and Dwinger[2]. Mukaidono[26] considers Problem 2. The papers Hajek, Bendova and Renc[27] and Cleave[28] contain material on the logical system associated with K.

With regard to the fuzzy switching function point of view, Zadeh's original paper in 1965 refers to Kleene[13]. The first work on Problem 3 in the literature is Preparata and Yeh[19].

Explicit descriptions of FK(n) were previously only known for $n = 1$, 2, and 3. For $n = 1$ the underlying lattice is shown in Fig. 1. For $n = 2$ the problem was considered in Preparata and Yeh[19] and $|\mathrm{FK}(2)| = 84$ is given. Berman and Dwinger[5] consider FK(2) as a lattice and also determine the cardinality and structure of it. Kandel[29] gives another method of enumerating FK(2). Cornish and Fowler[7] contains a complete listing of FK(2) and the algebraic structure of $\underline{K}$ is presented. The FORTRAN program in Berman and Wolk[51] may be used to explicitly construct the 84 district truth tables for solving Problem 2 for $n = 2$.

For $n = 3$, it is known that FK(3) has 43918 elements. This computation was done from the fuzzy switching function point of view by Mukaidono[31, 32]. Berman and Köhler[33] obtain the same result by use of a general program for distributive lattice.

2. FK(n)

Our method of describing FK(n) is quite simple. Define an equivalence relation on FK(n) by having p and q equivalent if and only if p and q are equivalent as Boolean

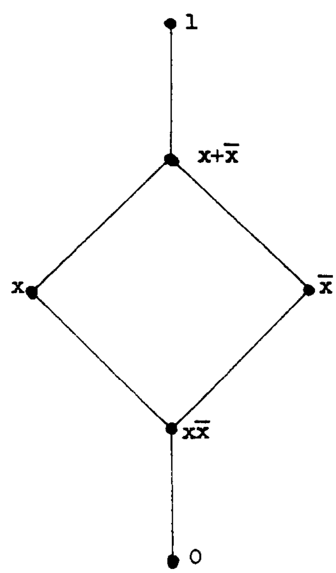

Fig. 1. The lattice for FK(1).

expressions. This equivalence relation has 2^{2^n} classes, and we then determine the structure and size of each such class.

Since the operations for algebras in $\underset{\sim}{K}$ are the same as those for Boolean algebras, every element of FK(n) has an interpretation as a Boolean expression. We wish to extend our method to algebras A for which three operations $+, \cdot, -$ are a subset of the operations of A and for which $\langle A;\ +,\ \cdot\rangle$ is a distributive lattice. This accounts for the following roundabout method of defining the relation on $\sim$ on FK(n). In Section 4 we consider such a more general situation.

In the free Boolean algebra on n free generators $x_1, \ldots, x_n$, a *minterm* is any polynomial of the form $x_1^{\epsilon_1} \ldots x_n^{\epsilon_n}$ where $\epsilon_i = 0$ or 1 and $x_i^0 = x_i$ and $x_i^1 = \bar{x}_i$. Let M denote this set of minterms. Note $|M| = 2^n$. It is known that in any free Boolean algebra, every element p is the unique sum of minterms $p = \Sigma\{m \mid m \in M, m \leq p\}$.

Define a relation $\sim$ on FK(n) by $p \sim q$ if and only if $\{m|m \in M,\ m \leq p\} = \{m|m \in M,\ m \leq q\}$. Note this implies $p \sim q$ if and only if p and q are equivalent as Boolean expressions. Let $B(p)$ denote the equivalence class of p with respect to $\sim$.

For any $p \in$ FK(n), $B(p)$ is a convex distributive sublattice of the lattice FK(n). We use some elementary lattice theory to describe $B(p)$.

In a finite lattice L, an element a is *join irreducible* if $a \neq 0$ and if $a = b + c$, then $a = b$ or $a = c$. In a partially ordered set (poset) P, a subset S of P is called an *antichain* if no two distinct elements of S are comparable.

The following two facts are found in say Birkhoff[34] or Grätzer[11].

Fact 1. Any finite distributive lattice is uniquely determined by its poset of join irreducibles.

Fact 2. The cardinality of a finite distributive lattice is equal to the number of antichains in its poset of join irreducibles.

Fact 3. Let L be a finite distributive lattice and let $a \leq b \in L$. The set of join irreducible elements of the interval sublattice $[a, b]$ is order isomorphic to the set of join irreducible elements z of L for which $a < a + z \leq b$.

Proof. If z is join irreducible in L and $a < a + z \leq b$, then it is easily seen that $a + z$ is join irreducible in the interval $[a, b]$. Also if $a + z = a + y$ with y, z join irreducible in L, then $z = az + yz$ so $z = yz$ and similarly $y = yz$, so $y = z$. Finally, let t be join irreducible in the lattice $[a, b]$, $t > a$. Write $t = a + z_1 + \ldots + z_k$ where each z_i is join irreducible in L. Regroup to obtain $t = (a + z_1) + (a + z_2) + \ldots + (a + z_k)$. So $t = a + z_i$ for some i.

The join irreducibles in the lattice FK(n) have been described by many authors; there is no standard nomenclature.

Fact 4. In FK(n) the join irreducible elements are of the following 3 distinct types:

type i $\quad x_{i_1}^{\epsilon_{i_1}} \ldots x_{i_k}^{\epsilon_{i_k}},\ 1 \leq i_1 < \ldots < i_k \leq n,\ 0 \leq k < n.$

type ii $\quad x_1^{\epsilon_1} \ldots x_n^{\epsilon_n} x_{i_1}^{1-\epsilon_{i_1}} \ldots x_{i_k}^{1-\epsilon_{i_k}},\ 1 \leq i_1 < \ldots < i_k \leq n, 0 < k \leq n.$

type iii $\quad x_1^{\epsilon_1} \ldots x_n^{\epsilon_n}.$

Note that there are $2 \cdot 3^n - 2^n$ join irreducibles in the lattice FK(n). Also, type iii join irreducibles are minterms. The element 1 of FK(n) is join irreducible and corresponds to the type i element with $k = 0$. The set of type i elements is dually order isomorphic to the set of type ii elements.

In order to describe $B(p)$ for any polynomial p it suffices to describe the poset of join irreducibles of the interval $B(p)$ in the lattice FK(n). The definition of $\sim$ implies that the least element of $B(p)$ is $\Sigma\{m|m \in M,\ m \leq p\}$. Denote this element by $\mu(p)$. So it remains to determine which join irreducible elements z of FK(n) have the property that $(z + \mu(p)) \sim p$ and $\mu(p) < z + \mu(p)$. Clearly z cannot be a minterm. If such a z is of type i, then it suffices that $\{m|m \in M,\ m \leq z\} \subseteq \{m|m \in M,\ m \leq p\}$. If z is of type ii, then z is equivalent to 0 as a Boolean expression. For such a z it suffices that $z \nleq \mu(p)$, which is equivalent to $\{m|m \in M,\ m \geq z\} \subseteq \{m|m \in M,\ m \nleq p\}$.

In the poset of join irreducibles of $B(p)$, those elements corresponding to join irreducibles z of type i are order unrelated to those corresponding to type ii join irreducibles. It follows that $B(p)$, as a lattice, is the direct product of two sublattices; for details see Birkhoff ([34], p. 57).

We summarize this method of determining $B(p)$ by the following algorithm.

Given a Boolean expression p, find the distributive lattice $B(p)$ contained in FK(n):

(1) List all minterms $m_1, m_2, \ldots, m_k$ with $m_i \leq p$.

(2) List those join irreducibles z of type i such that $z \not\geq m$ for any minterm m, $m \notin \{m_1, \ldots, m_k\}$.

(3) Determine the distributive lattice L_1 formed by the poset of join irreducibles in step 2.

(4) List those join irreducibles z of type ii such that $z \not\leq m$ for any minterm m, $m \in \{m_1, \ldots, m_k\}$.

(5) Determine the distributive lattice L_2 formed by the poset of join irreducibles in step 4.

(6) $B(p)$ is isomorphic to $L_1 \times L_2$. In particular, $|B(p)| = |L_1| \times |L_2|$.

The algorithm given above can be used to describe FK(n) by simply applying it to all p in the free Boolean algebra on n generators, FB(n). There are 2^{2^n} such elements. A substantial simplification of this can be acheived by partitioning FB(n) into classes of Boolean polynomial that are N^G equivalent: that is, they are equivalent modulo complementations and permutations of variables and function negation. Harrison[35] contains a detailed exposition of N^G equivalence. Some authors call this NPN-equivalence, e.g., Muroga[36]. If p and q in FB(n) are in the same N^G class, then $B(p)$ and $B(q)$ are clearly isomorphic as lattices. Thus, the computation of $B(p)$ may be confined to a single p from each N^G class. For $n = 2$, there are 4 classes: 0, xy, $xy + x\bar{y}$, $x\bar{y} + \bar{x}y$. For $n = 3$ there are 14 classes and for $n = 4$ there are 222 classes.

For $n = 3$ we give the complete analysis: The poset of all type i join irreducibles is dually isomorphic to the poset of type ii join irreducibles. We draw the 19 elements of type ii in Fig. 2; Table 1 contains a tabulation for FK(3).

In Table 1, the left column is the number of the polynomial as given in the Harvard[37] table; the next column is a list of minterms m, $m \leq p$. The column headed MULT gives the cardinality of the N^G class for p. Then comes the list of relevant join irreducibles for each type and the number of antichains in each poset. The final column is the total contribution of eachN^G class. It is the product of the column headed MULT, and the two columns with the number of antichains.

3. FK(4)

We outline our work on FK(4). Hindsight shows $|FK(4)| > 10^{11}$ and so an enumeration, counting elements one at a time, is not feasible on today's computers. Therefore the method, as in Berman and Köhler[33], of counting the elements one at a time will not work here.

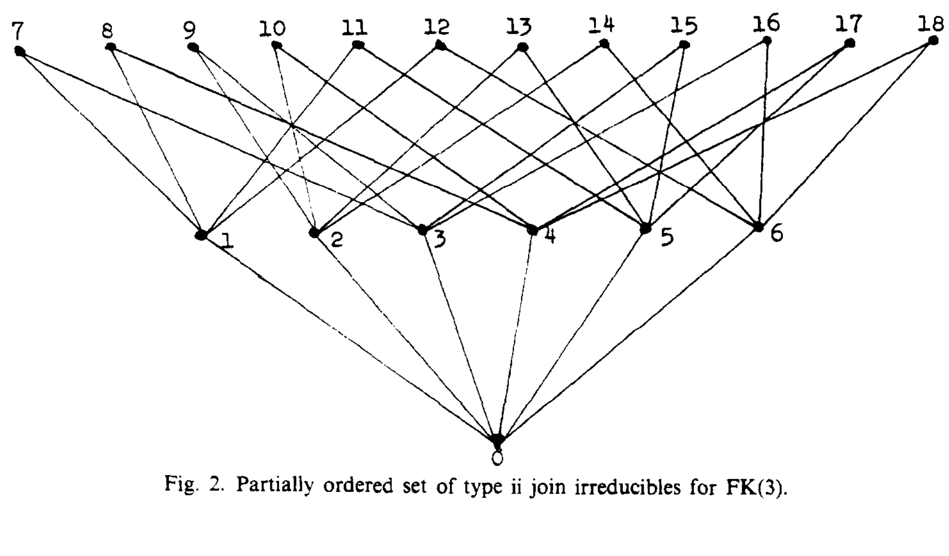

Fig. 2. Partially ordered set of type ii join irreducibles for FK(3).

Table 1.

NAME	MINTERM LIST	MULT	TYPE i ELEMENTS	NO. ANTI CHAINS	TYPE ii ELEMENTS	NO. ANTI CHAINS	TOTAL
0	NONE	2	NONE	1	ALL	6211	12422
1	0	16	NONE	1	1,3,5,7,8,9,11, 12,13,15,16,17	621	9936
2	0,7	8	NONE	1	8,9,12,13,16,17	64	512
3	0,3	24	NONE	1	1,7,8,11,12,16,17	68	1632
4	0,1	24	10	2	1,3,7,8,9,11,12,15,16	145	6960
5	1,2,4	16	NONE	1	7,11,15	8	128
6	0,3,4	48	18	2	7,11,16,17	16	1536
7	0,1,2	48	10,14	4	1,7,8,11,12,15	34	6528
8	0,1,2,3	6	2,9,10,13,14	17	1,7,8,11,12	17	1734
9	0,1,2,4	8	10,14,18	8	7,11,15	8	512
10	0,1,4,6	24	10,12,18	8	9,11,15	8	1536
11	1,2,3,4	24	9,13	4	7,11	4	384
12	1,2,4,7	2	NONE	1	NONE	1	2
13	0,3,4,7	6	15,18	4	16,17	4	96
		256					43918

In order to implement the algorithm described in the previous section we wrote a computer program which accepts as input data information about the 222 distinct N^G classes of Boolean polynomials, and a table of join irreducibles of FK(4). The program then computes the join irreducibles in $B(p)$ for each of the 222 representative expressions p. Then the number of antichains in each poset is found. $|\mathrm{FK}(4)|$ is the sum of these 222 numbers: $|\mathrm{FK}(4)| = 160{,}297{,}985{,}276$. This was announced in Berman and Mukaidono[30].

Some aspects of this computation merit comment. A readily available source of the 222 classes is in the Appendix to Harrison[35]. However, Harrison's list does not contain the cardinality of each class. The Harvard tables, Harvard[37], do contain these numbers; however the numbering here differs from Harrison's. Another source of the necessary information on the 222 classes is in de Troye[38]. (Our work uncovered a misprint in this table: the function numbered 141 on page 263 should have 384 in the column headed p.)

The poset of join irreducibles of FK(n) is self-dual. The set of type i elements is dually isomorphic to those of type ii. This allows for a systematic coding of steps 2 and 4 of the algorithm using one poset of $3^4 - 2^4 = 65$ elements.

The enumeration of the antichains in each poset is done by a backtracking program, as in, say, Berman and Köhler[33]. Of the 222 classes, 221 can be easily handled by such a program. One class however, the class of the constant 0, is too large for such a program—it has 31, 901, 034, 831 elements. But since it has only length 4 as a poset, further simplifications in enumerating its antichains are possible, thereby allowing for the determination of $|\mathrm{FK}(4)|$.

Each author performed the computation: a FORTRAN program on PDP-11 was used at Meiji University and a PL/I program on an IBM/370 was used at the Computer Center of the University of Illinois. The entire computation took 2.5 hr on the IBM machine. We acknowledge the assistance of H. Aikawa, N. Rickert, and H. Tatsumi. A table describing the 222 classes for FK(4), analogous to Table 1 for FK(3), may be obtained from the authors.

4. ANOTHER EXAMPLE

The method of enumerating a free algebra with respect to the equivalence classes of $\sim$ may be used for any algebraic system which includes two binary operations that form a distributive lattice, which has a nontrivial unary operation, and for which the join irreducible elements of this distributive lattice can be described. Of course if the join irreducibles are known, then the distributive lattice can always be determined—the method has value only in those cases where the number of join irreducibles is so large as to make such a direct description impractical.

An example of this is to consider the 3 element algebra K in which all the elements are designated as constants. Let $R = < \{0, h, 1\}; +, \cdot, -, 0, h, 1 >$ denote this algebra, and let $\underset{\sim}{R}$ be the equational class generated by it. Problem 1 then becomes that of describing the free algebra FR(n). This is also the same as finding all truth tables as in Problem 2, but with the use of all three constant truth tables allowed. The truth tables arising in this case are called *regular* in Kleene[13]. The actual definition of regular given by Kleene is different, the equivalence to the definition in terms of the algebra R is due to Mukaidono[39]. The class $\underset{\sim}{R}$ and FR(n) are also studied in the Russian work on systems of algorithmic languages, e.g. Tseitlin[40].

At first glance it would seem that adding the constant function h to the algebra K would have little effect on the equational theory. However, as we shall see, this radically alters the algebra FR(n). Moreover, from the point of view of switching functions, it is reasonable to allow the three constant functions $0, h, 1$. Thus the class $\underset{\sim}{R}$ is a natural one to study in this context.

In FR(n), as in FK(n), define a minterm to be an element of the form $x_1^{c_1} \ldots x_n^{c_n}$. Let M denote the set of all minterms. Since R has a distributive lattice structure, the relation $\sim$ may be defined on FR(n): $p \sim q$ if and only if $\{m | m \in M,\ m \leq p\} = \{m | m \in M,\ m \leq q\}$.

Since h is a constant, algebras in $\underset{\sim}{R}$ do not contain Boolean algebras as subalgebras or as homomorphic images. Also, $\underset{\sim}{R}$ satisfies $\bar{h} = h$ and $x\bar{x} \leq h \leq y + \bar{y}$ for arbitrary variables x and y.

LEMMA

The join irreducibles of FR(n) are of four types:

type i: $x_{i_1}^{c_{i_1}} \ldots x_{i_k}^{c_{i_k}},\ 1 \leq i_1 < \ldots < i_k \leq n,\ 0 \leq k < n.$

type ii: $x_1^{c_1} \ldots x_n^{c_n} x_{i_1}^{1-c_{i_1}} \ldots x_{i_k}^{1-c_{i_k}},\ 1 \leq i_1 < \ldots < i_k \leq n,\ 0 < k \leq n.$

type iii: $x_1^{c_1} \ldots x_n^{c_n}.$

type iv: $x_1^{c_1} \ldots x_n^{c_n} h.$

Proof. It is easily verified that any join irreducible in FK(n) is also join irreducible in FR(n). Any other join irreducible in FR(n) must be of the form zh where z is join irreducible in FK(n). For any variable x_i, $zh = zh(x + \bar{x}_i) = zhx_i + zh\bar{x}_i$. So if zh is join irreducible, then z must contain x_i or $\bar{x}_i$ as a factor, for all i. Also, $h \geq z$ for any type ii join irreducible of FK(n). So if zh is join irreducible, then z must be a minterm and so zh is of type iv. By considering mappings into R it is easily seen that type iv elements are indeed join irreducible.

It follows that FR(n) has $2 \cdot 3^n$ join irreducible elements. Hence for small n, FR(n) can be easily discribed: FR(0) is the 3 element chain, FR(1) has 11 elements as in Fig. 3. FR(2) has 197 elements; the poset of join irreducibles is in Fig. 4.

For larger n we use the method of Section 2 to describe FR(N). Let p be any Boolean polynomial and let $S = \{m | m \in M,\ m \leq p\}$. We wish to determine the poset of join irreducibles for the interval sublattice $B(p) = \{q | q \in \mathrm{FR}(n),\ q \sim p\}$. Let $\mu(p) = \Sigma S$ be the least element of $B(p)$. The poset of join irreducibles of $B(p)$ is isomorphic to the set of all join irreducibles $z \in \mathrm{FR}(n)$ for which $(z + \mu(p)) \sim p$ and $\mu(p) < z + \mu(p)$. Such z of type iv are any elements mh where $m \in M \backslash S$.

Define a relation $\sim_1$ on $B(p)$ by $q \sim_1 r$ if and only if $\{m | m \in M,\ mh \leq q\} = \{m | m \in M,\ mh \leq r\}$. It is easily seen that the $\sim_1$ equivalence class of any $q \in B(p)$ is an interval sublattice of $B(p)$. The join irreducibles of type i in the $\sim_1$ equivalence class for $q \in B(p)$ are all z of type i such that $z \ngeq m$ for any $m \in M \backslash S$. Note this depends only on p, and is

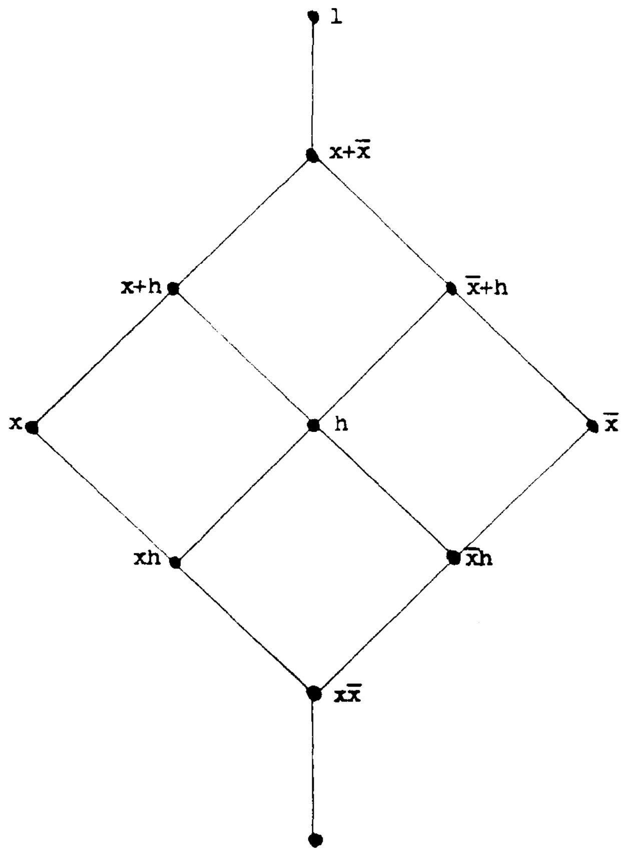

Fig. 3. The lattice for FK(1).

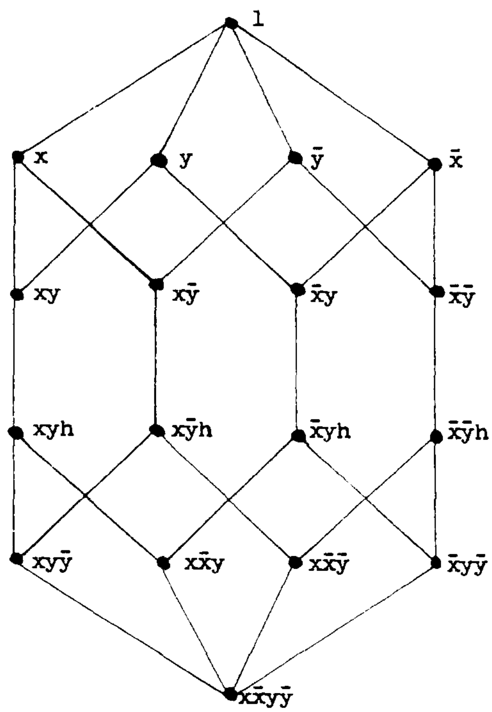

Fig. 4. Partially ordered set of join irreducibles for FR(2).

independent of q. The set of all such type i join irreducibles for p is the same as in FK(n). For $q \in B(p)$, let $T = \{m | m \in M \backslash S, mh \leq q\}$. Note $S \cap T = \emptyset$. The join irreducibles of type ii for the $\sim_1$ equivalence class of $q \in B(p)$ are all z such that $z \nleq m$ for any $m \in S \cup T$. This poset is the same as the poset of type ii join irreducibles for the sublattice $B(\Sigma(S \cup T))$ of FK(n). So the structure of $B(p)$ in FR(n) can be completely determined from that of the $\sim$ classes of FK(n).

For $n = 3$ and 4, this description of FR(n), by use of FK(n), can be accomplished. By using N^G equivalence to reduce the number of cases, and by using rapidly computed invariants (e.g., Jackson and Ankerlin[41]) to identify the N^G class of an arbitrary polynomial, the computation can be efficiently performed.

THEOREM

FR(3) has 129,615 elements. FR(4) has 430,904,428,717 elements.

The computations were done independently by both authors. The total computation time on an IBM/370 for FR(4) was 15 min.

5. UPPER AND LOWER BOUNDS

There have been many attempts to obtain upper and lower bounds and asymptotic estimates for the cardinality of FK(n): Kandel[29], Kameda and Sadeh[42], Schwede and Kandel[43], Kandel and Clark[44], Mukaidono[45]. The sharpest general results so far are in Thum and Kandel[46]

$$2^{2^{l}\binom{n}{l}} < |\mathrm{FK}(n)| < 2^{3^n} \tag{1}$$

where l is the value of k which maximizes $2^k\binom{n}{k}$. It is easily seen that $l = \ulcorner 2n/3 \urcorner$. By using Stirling's approximation for factorials it can be shown (e.g. Thum and Kandel[46]) that there exist constants c_2 and c_1 such that for n large,

$$\frac{3}{2\sqrt{\pi}}\frac{3^n}{\sqrt{n}}\left(1+\frac{c_2}{n}\right) \leq 2^l\binom{n}{l} \leq \frac{3}{2\sqrt{\pi}}\frac{3^n}{\sqrt{n}}\left(1+\frac{c_1}{n}\right) \tag{2}$$

The use of (2) in (1), allows a comparison of the two bounds in (1). We sharpen (1) to

$$2^{2^l\binom{n}{l}}\epsilon^{\binom{n}{l+1}} < |\mathrm{FK}(n)| < 2^{2^n} n^{2^l\binom{n}{l}} \tag{3}$$

where $\epsilon \to e$ as $n \to \infty$. By using (2) in (3), the upper bound in (3) can be used to give

$$|\mathrm{FK}(n)| < 2^{3^n\left(\left(\frac{2}{3}\right)^n + \frac{3\log_2(n)}{2\sqrt{\pi n}}\left(1+\frac{c_1}{n}\right)\right)} \tag{4}$$

which improves the upper bound in (1).

If P is any poset, let $\alpha(P)$ denote the number of antichains in P. Let $P(n)$ denote the poset of type i join irreducibles in FK(n). If p is any n-variable Boolean polynomial, and if $B(p)$ is the sublattice of FK(n) consisting of all $q \in$ FK(n) such that $p \sim q$, then the results of Section 2 show that $|B(p)| \leq \alpha(P(n))$. Since $|P(n)| = 3^n - 2^n$ and there are 2^{2^n} n-variable Boolean polynomials, it follows that $|\mathrm{FK}(n)| < 2^{2^n} 2^{3^n - 2^n} = 2^{3^n}$. This gives the upper bound in (1).

We now improve this upper bound. $P(n)$ is a ranked poset having $2^k\binom{n}{k}$ elements at the kth rank, $k = 0, 1, \ldots, n-1$. Here we count rank from the top, so the maximal element of $P(n)$ has rank 0. By Baker[47], $P(n)$ enjoys the Sperner property: the largest sized antichain in $P(n)$ has cardinality m where $m = \max_k \{2^k\binom{n}{k}\}$. Dilworth's theorem (Dilworth[48] or Birkhoff[34], pp. 98–99) shows that $P(n)$ is the disjoint union of m chains. If A is any antichain in $P(n)$, then A is uniquely determined by its intersection with each of these chains. The set A intersects any chain in at most one element. We can safely ignore the maximal element of $P(n)$ and so each chain has at most $n - 1$ elements. There are at most n possibilities for the intersection of A with any chain. So $\alpha(P(n)) \leq n^m$. Using $m = 2^l\binom{n}{l}$ where $l = \lceil 2n/3 \rceil$ gives the upper bound in (3). The bound in (4) is obtained from (3) using the upper bound in (2).

This arument for an upper bound on $\alpha(P(n))$ will work for any finite poset satisfying the Sperner property. It was used in Gilbert[49] for estimating the size of the free distributive lattice on n generators.

In order to obtain a lower bound, note that for any k, the $2^k\binom{n}{k}$ elements of rank k form an antichain. It follows that $2^{2^k\binom{n}{k}} \leq |\mathrm{FK}(n)|$. Choosing l to maximize $2^k\binom{n}{k}$ gives the lower bound of (1). We will sharpen this bound by counting antichains in two adjacent ranks of $P(n)$. This is similar to Shapiro[50].

In the kth rank, each element is covered by k elements of rank $k-1$. So r elements of rank k are covered by at most rk elements of rank $k-1$. So the number of antichains in these two ranks is at least

$$\sum_r \binom{2^k\binom{n}{k}}{r} 2^{2^{k-1}\binom{n}{k-1} - rk} = 2^{2^{k-1}\binom{n}{k-1}} \sum_r \binom{2^k\binom{n}{k}}{r} 2^{-rk}.$$

Applying the binomial theorem gives

$$2^{2^{k-1}\binom{n}{k-1}}(1 + 2^{-k})^{2^k\binom{n}{k}} = 2^{2^{k-1}\binom{n}{k-1}} \epsilon_k^{\binom{n}{k}}$$

where $\epsilon_k \to e$ as $k \to \infty$. Finally letting $l = k-1$ gives the lower bound in (3).

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